Simultaneous Embedding with Two Bends per Edge in Polynomial Space

Frank Kammer
Institut für Informatik, Universität Augsburg, D-86135 Augsburg, Germany
kammer@informatik.uni-augsburg.de

Abstract

The simultaneous embedding problem is, given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, to find planar embeddings $\varphi(G_1)$ and $\varphi(G_2)$ such that each vertex $v \in V$ is mapped to the same point in $\varphi(G_1)$ and in $\varphi(G_2)$. This article presents a linear-time algorithm for the simultaneous embedding problem such that edges are drawn as polygonal chains with at most two bends and all vertices and all bends of the edges are placed on a grid of polynomial size. An extension of this problem with so-called fixed edges is also considered.

A further linear-time algorithm of this article solves the following problem: Given a planar graph G and a set of distinct points, find a planar embedding for G that maps each vertex to one of the given points. The solution presented also uses at most two bends per edge and a grid polynomial in the size of the grid that includes all given points. An example shows two bends per edge to be optimal.

1 Introduction

The visualization of information has become very important in recent years. The information is often given in the form of graphs, which should at the same time aesthetically please and convey some meaning. Many aesthetic criteria exist, such as straight-line edges, few bends, a limited number of crossings, depiction of symmetry and a small area of the drawing given, e.g., a minimal distance between two vertices. If graphs change over the course of time or if different relations among the same objects are presented in graphs, it is often useful to recognize the features of the graph that remain unchanged. If each graph is drawn in its own way, in other words if the graphs are embedded independently, there is probably only little correlation. Therefore, the embeddings of the graphs have to be constructed simultaneously to achieve that all or at least some features of the graph are fixed.

A viewer of a graph quickly develops a mental map consisting basically in the positions of the vertices. If k planar graphs with the same vertex set V are presented, it is desirable that the positions of all vertices in V remain fixed. This problem is called *simultaneous embedding*. An extension of the problem is the so-called *simultaneous embedding with fixed edges*: In addition to the k graphs, a set of edges F is given. A feasible solution is an embedding of the k graphs such that all vertices and all edges in F have fixed embeddings.

An algorithm for the simultaneous embedding problem for k planar graphs with few bends per edge helps to find an embedding with few bends per edge for graphs of *thickness* k. The thickness of a graph G is the minimum number of planar subgraphs into which the edges of G can be partitioned. Since a graph of thickness k can be embedded in k layers without any edge crossings, thickness is an important concept in VLSI design. Additionally, an algorithm for simultaneous embedding of k planar graphs with fixed edges helps to find an embedding of a graph of thickness k such that certain sets of edges are drawn straight-line as well as identically in all layers.

Definition 1 *A k-bend embedding of $G = (V, E)$ is an embedding such that each edge in E is drawn as a polygonal chain with $\leq k$ bends. Thus, an edge with l bends consists of $l + 1$ straight-line segments.*

Unless stated otherwise, the following embeddings place all vertices and all bends on a grid of size polynomial in the number of vertices. According to results of Pach and Wenger [8], for any number of planar graphs on the same vertex set of size n, an $O(n)$-bend simultaneous embedding is possible. Erten and Kobourov [5] show with a small example that a 0-bend simultaneous embedding does not always exist for two planar graphs. They show that three bends suffice to embed two general planar graphs and that one bend is enough in the case of two trees. By using a new algorithm presented in Section 3, this article shows in Section 2 that the number of bends per edge in a simultaneous embedding of two planar graphs can be reduced to two.

Erten and Kobourov also examine simultaneous embeddings with fixed edges in the special case where one input graph is a tree and the other is a path. For special kinds of graphs (caterpillar and outerplanar graphs), Brass et al. [1] show how to embed simultaneously two of the special graphs such that all edges are fixed. For general graphs, the simultaneous embedding problem with fixed edges is considered in Section 4. If all edges fixed, this problem is already for almost all instances of two planar graphs not solvable (Section 5)—even if the number of bends per edge is unbounded. Therefore, the algorithm presented in Section 4 works only with sets of fixed edges with certain properties.

Kaufmann and Wiese [6] present an algorithm for the *vertices-to-points* problem, which computes an embedding of a planar graph such that the vertices are drawn on a grid at given points. If all vertices and all bends are placed on a grid whose size is polynomial in the size of the grid of the given points, their embedding requires up to three bends per edge, but via a similar algorithm as for the simultaneous embedding problem, a 2-bend embedding can be constructed (Sections 2 and 3). If a outer face is specified, Kaufmann and Wiese show that an 1-bend embedding for the vertices-to-points problem is not possible in general. In section 5, a very short proof of the same lower bound is presented such that no outer face must be specified.

2 Finding an embedding

Since the same ideas as already described in [6, 1, 5] are used, these will only be sketched. Many parts of the ideas help to find a k-bend embedding for a small k for both of the two problems below. Assume for the time being that for all planar graphs $G = (V, E)$ considered in the following, a Hamilton cycle C exists and is known. Moreover, let f_G be a bijective function that maps each vertex to a number in $\{1, \ldots, |V|\}$ such that consecutive vertices in C have consecutive numbers modulo $|V|$. The knowledge of the Hamilton cycle C is useful because in a planar embedding of G, each edge not part of C is either completely inside or completely outside C. In the following two problems are defined and their solutions are presented subsequently.

Definition 2 *The simultaneous embedding problem is, given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, to find planar embeddings $\varphi(G_1)$ and $\varphi(G_2)$ such that all vertices are fixed, i.e. $\forall v \in V : \varphi_1(v) = \varphi_2(v)$.*

Observe that each vertex v is associated with two numbers x, y, where $x = f_{G_1}(v)$ and $y = f_{G_2}(v)$. As a first step to embed G_1 and G_2, use these two numbers of each vertex as its coordinates. Embed the edges in G_1 and G_2 by applying the procedure described after the following definition once for G_1 with $direction = horizontal$ and once for G_2 with $direction = vertical$.

Definition 3 *Let $G = (V, E)$ be a planar graph and let P be a set of distinct points in the plane. The vertices-to-points problem is to find a planar embedding φ such that $\forall v \in V : \varphi(v) \in P$.*

For an embedding, sort the given points according to their x-coordinates. Map the vertex v with number $i = f_G(v)$ to the point with the i-th smallest x-coordinate. Continue the embedding of the edges with $direction = horizontal$.

In the following the procedure to embed the edges is described:

Denote the graph under consideration by $G = (V, E)$ and the edge $\{f_G^{-1}(1), f_G^{-1}(|V|)\}$ by $\hat{e}$. W.l.o.g. assume that $direction = horizontal$. Otherwise turn around the construction by 90 degree.

First, embed the edges of the Hamilton path $P = C \setminus \{\hat{e}\}$ as straight lines. For each edge $e \in P$ let x_e and y_e be the absolute values of the differences of the x- and y-coordinates of the endpoints of e. Set $\alpha = \min_{e \in P} \tan(x_e/y_e)$. For each vertex v, let l_v be the vertical line through v. Using a combinatorial embedding of G, partition the edges not part of C in linear time into two sets E_1 and E_2 such that each set can be embedded inside (or outside) the Hamilton cycle without edge intersections. Add the edge $\hat{e}$ to E_1, say. Embed each edge $\{u, v\}$ in E_1 below P and in E_2 above P as part of two rays starting from vertex u to the right of l_u and from vertex v to the left of l_v, when $f_G(u) < f_G(v)$. Draw each ray in such a way that the angle between the ray and the corresponding vertical line is α and cut off the two rays at their point of intersection. If a vertex has several incident edges embedded on the same side of P or if the point of intersection is not on the grid, modify the angle slightly such that planarity is preserved. This yields a 1-bend embedding of G.

However one problem remains: How to find a Hamilton cycle and what to do if no Hamilton cycle exists. The solution is to modify G. According to Chiba et al. [2], G can be made 4-connected preserving planarity by repeated applying

Operation 1: adding an edge and

Operation 2: splitting an original edge of G once and adding a new vertex between the two parts of the split edge.

Denote this modified graph by G'. In [3], Chiba et al. show that every 4-connected graph has a Hamilton cycle that can be found in linear time. Use an embedding for G' to obtain an embedding for G by removing the new edges, merging the embeddings of the two parts of each split edge and replacing each new vertex by a bend for the corresponding edge.

Observe that an edge $e = \{v_1, v_2\}$ in G corresponds to at most two split edges $e_1 = \{v_1, v_{\text{new}}\}$ and $e_2 = \{v_{\text{new}}, v_2\}$ in G'. If both edges e_1, e_2 are embedded with one bend and there is a further bend between the edges e_1, e_2 at v_{new}, the edge e is embedded with three bends. As we see later, one part of the two split edges is inside and the other part is outside of the Hamilton cycle used. Thus, this third bend at v_{new} exists only if v_{new} does not appear between v_1 and v_2 in the Hamilton path used for the embedding.

Using a shrinking angle during the process of embedding instead of an almost fixed angle α, Kaufmann and Wiese described in [6] how to remove the bend point at v_{new}, but this solution requires a grid of exponential size to place the bends of the edges.

Since it is essential where the numbering along the Hamilton cycle starts, let us consider the problem of finding a so-called *closable* Hamilton path. A Hamilton path is closable if it is contained in a Hamilton cycle. A closable Hamilton cycle makes it more explicit which part of the Hamilton cycle is used to number the vertices.

Definition 4 *An edge-extension of a graph G is a graph G^+ obtained from G by adding auxiliary edges or by splitting edges, i.e. replacing each split edge by a path of length two whose midpoint is a so-called* new *vertex of degree 2. Thus, each edge in G corresponds to a unique path in G^+ of arbitrary length.*

By the use of Operations 1 and 2, a new and simple linear-time algorithm for the problem of finding a closable Hamilton path in an edge-extension G^+ of the given graph G is presented in the next section such that each edge in G corresponds to a path of length ≤ 2 in G^+. Moreover, the constructed closable Hamilton path has the *between property*:

Definition 5 *Let G^+ be an edge-extension of $G = (V, E)$ and let P be a simple path in G^+. P has the between property (in G^+ with respect to G)*

if each new vertex that was inserted between the two split parts of an edge $\{u, v\}$ is between u and v on the simple path P.

From the considerations, we can conclude the following.

Theorem 6 *Given two planar graphs G_1 and G_2 on the same vertex set of size n, a 2-bend simultaneous embedding of G_1 and G_2 with area $n^{O(1)}$ can be found in $O(n)$ time.*

Theorem 7 *Given a planar graph G and a set of distinct points P on a grid, a 2-bend embedding of G can be found in linear time such that each vertex is embedded to a point in P and such that the area of the embedding of G is polynomial in the size of the grid of the given points.*

3 Finding a closable Hamilton path

An extension H of G is first constructed. Although H will not be planar, a closable Hamilton path in H helps to construct a closable Hamilton path in a planar extension of G. Obtain $G' = (V, E)$ by triangulating G. Denote by $\varphi(G')$ a fixed combinatorial embedding of G' and choose an arbitrary face of φ to be the outer face. Let $G'_D = (W, F)$ be the dual graph of G' without having a vertex (and its edges) for the outer face. For each vertex $w \in W$ representing a face A of $\varphi(G')$, denote by $\Delta(w)$ the set of the three vertices adjacent to A. Define $D = \{(u, v) \mid u \in W \wedge v \in \Delta(u)\}$ and $H = (V \cup W, E \cup F \cup D)$. See Figure 1 for an example, but for the time being ignoring the distinction between vertices inside and outside the set A_{i-1}.

Define an area as the union of some faces of $\varphi(G')$ including all adjacent vertices and edges. For an area A, let $V_A \subseteq V \cup W$ be the set of vertices in A, let $V_A^- \subseteq V \cap V_A$ be the set of vertices on the border of A adjacent to a vertex in $V \setminus V_A$ and let $E_A^- \subseteq E$ be the set of edges on the border of A. Observe that not every vertex on the border of A is part of V_A^-. Choose $\hat{e} = \{v_1, v_2\} \in E$ as an arbitrary edge adjacent to the outer face of $\varphi(G')$ and let $w \in W$ be the vertex of the dual graph that corresponds to the inner face of $\varphi(G')$ adjacent to $\hat{e}$. Moreover, denote the area of this inner face by A_0 and let v_3 be the third vertex adjacent to this inner face (i.e. $\Delta(w) = \{v_1, v_2, v_3\}$). Thus, $V_{A_0} = \{v_1, v_2, v_3, w\}$, $V_{A_0}^- \subseteq \{v_1, v_2, v_3\}$ and $E_{A_0}^- = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$.

Using $P_0 = (v_2, v_3, w, v_1)$ as a first simple path in H and A_0 as the processed area, the aim is to extend P_0 and A_0 stepwise such that the following invariants are true after each step i for the processed area A_i and the current path P_i:

Invariant 1: P_i is a simple path containing all vertices in V_{A_i}.

Invariant 2: For all edges $\{u, v\} \in E$ that are crossed by P_i using an edge $e_{\{u,v\}}$ in F, the sub-path of P_i connecting u and v contains $e_{\{u,v\}}$.

Invariant 3: The vertices in P_i are in the same order in P_i and on the border of A_i, starting with v_2.

Invariant 4**:** For all edges $(u, v) \in E^-_{A_i}$ one of the following is true:

Property a**:** (u, v) is part of the current path P_i.

Property b**:** Let $w \in W$ be the vertex corresponding to the face of G' that is adjacent to (u, v) and inside the processed area A_i. Either (u, w) or (v, w) is part of current path P_i.

These invariants are all true for P_0 and A_0. Initially $(i = 0)$ and in each step i, calculate the sets $V_{A_i}, V^-_{A_i}, E^-_{A_i}$ and for each vertex v the list $V^v_{A_i} = \{u \in V \,|\{v, u\} \in E \wedge |\{v, u\} \cap V_{A_i}| = 1\}$ ordered in counter-clockwise order around v in $\varphi(G')$. This list contains all vertices adjacent to v that are with respect to v on the opposite side of A_i.

If step i adds a vertex $s \in V$ to the processed area, all these sets and lists can be updated in time *O(number of vertices adjacent to s)*.

Step i is carried out as follows: Choose $s \in V^v_{A_{i-1}}$ for some vertex $v \in V^-_{A_{i-1}}$ on the border of A_{i-1}. While only one vertex is to be added to the processed area, test if the processed area together with the edges from s to vertices in $V^s_{A_{i-1}}$ encloses additional vertices $t \in V \setminus \big(V_{A_{i-1}} \cup \{s\}\big)$. If such a vertex t exists, put s on a stack and process t first. This test if such

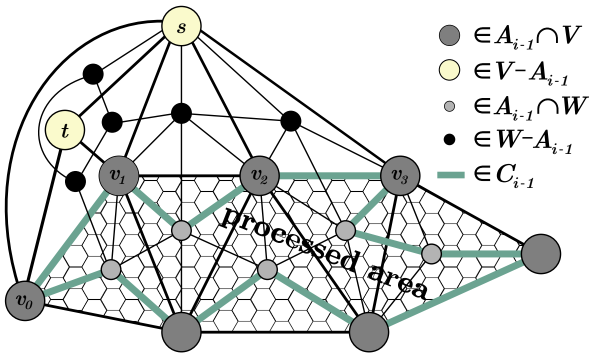


Figure 1: Extended graph H of a graph $G' = (V, E)$

a vertex t exists is easy: Let $v_0, \dots, v_k$ be the vertices of the ordered list $V^s_{A_{i-1}}$. In other words, these vertices are all adjacent to s and they appear in clockwise order on the border of A_{i-1}. See Figure 1. Consider in $\varphi(G')$ the vertices adjacent to s in counter-clockwise order from v_0 to v_k. If these vertices are all in $V^s_{A_{i-1}}$, no such vertex t exists. Otherwise choose t as the first vertex found that does not belong to $V^s_{A_{i-1}}$. After processing t, continue this check for s. If no such vertex t exists (any more), the $k+1$ vertices in $V^s_{A_{i-1}}$ together with s define k faces $W_s = \{w_1, \dots w_k\}$. Number these faces such that w_j is adjacent to v_{j-1} and v_j. In other words, each vertex $w \in W_s$ is adjacent in H to s and to two vertices in $V^-_{A_{i-1}}$. Extend the processed area A_{i-1} by the faces in W_s. For calculating the simple path P_i, two cases are considered. Figure 2 gives an illustration of case 1, Figure 3 of case 2.

Case 1. $\exists w_j \in W_s$: An edge $\{v_{j-1}, v_j\}$ in P_{i-1} is part of the border of the face w_j. Set

$$\begin{aligned} P_i &= (P_{i-1} \setminus \{v_{j-1}, v_j\}) \\ &\cup \{\{v_j, w_{j+1}\}, \{w_{j+1}, w_{j+2}\}, \ldots, \{w_{k-1}, w_k\} \{w_k, s\}\} \\ &\cup \{\{s, w_1\}, \{w_1, w_2\}, \ldots, \{w_{j-1}, w_j\}, \{w_j, v_{j-1}\}\}. \end{aligned}$$

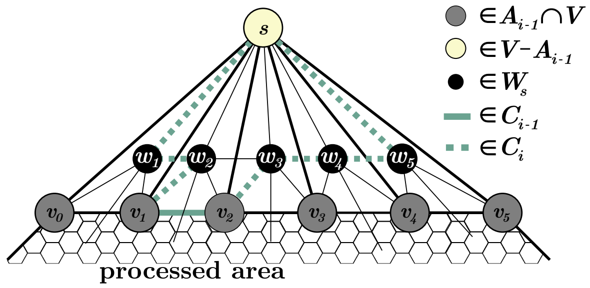


Figure 2: Face w_2 is adjacent to an edge in P_{i-1}

Case 2. Otherwise, let $\hat{w} \in W \cap V_{A_{i-1}}$ be the vertex inside the processed area A_{i-1} adjacent to v_0 and v_1. Since property a of Invariant 4 does not hold, we can assume that $\{v_0, \hat{w}\} \in P_{i-1}$ or $\{v_1, \hat{w}\} \in P_{i-1}$. In the first case set $\hat{P} = \{\{v_0, s\}, \{w_1, w_2\}\}$ and $\hat{v} = v_0$; in the other case set $\hat{P} = \{\{w_1, s\}, \{v_1, w_2\}\}$ and $\hat{v} = v_1$. Then

$$\begin{aligned} P_i &= (P_{i-1} \setminus \{\hat{w}, \hat{v}\}) \cup \{\{\hat{w}, w_1\}\} \cup \hat{P} \\ &\cup \{\{w_2, w_3\}, \ldots, \{w_{k-1}, w_k\}, \{w_k, s\}\}. \end{aligned}$$

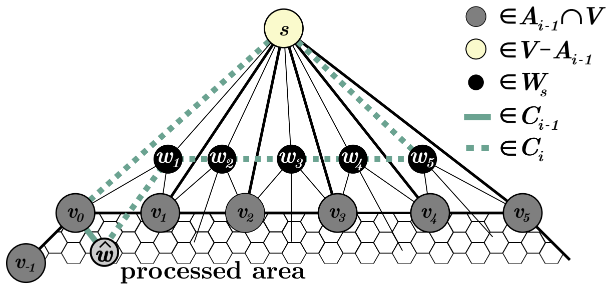


Figure 3: No face in W_s is adjacent to an edge in P_{i-1}

By the construction of P_i and by Invariant 3 of the last step, Invariants 1 and 2 are true after the i-th step. Since the border of A_i results from the border of A_{i-1} by a replacement of $v_0, \ldots, v_k$ by s and since the simple path P_i is an extension of P_{i-1} such that s is inserted between some vertices in $\{v_0, \ldots, v_k\}$, Invariant 3 is preserved.

Observe that for each edge in $E^-_{A_i} \setminus E^-_{A_{i-1}}$, either Property a or Property b of Invariant 4 is true. Furthermore, in Case 1, the edge $\{v_{j-1}, v_j\} \in P_{i-1} \setminus P_i$ is not in $E^-_{A_i}$ any more after step i. In Case 2, let $v_{-1} \in \Delta(\hat{w}) \setminus \{v_0, v_1\}$. If $\{v_{-1}, v_0\} \in E^-_{A_{i-1}}$, then v_0 is adjacent to only three vertices in A_{i-1} and thus $\{v_{-1}, v_0\} \in P_{i-1}$. Altogether, Invariant 4 is also true after the i-th step.

After $|V| - 3$ steps, $A_{|V|-3}$ equals to the whole internal area of G'. Because of Invariant 1, a closable Hamilton path $P_{|V|-3}$ in H has been found. It remains to show how to use the knowledge of a closable Hamilton path in H to find a closable Hamilton path P in a planar extension of G' that is also a planar extension of G. Let $v_{\sigma_1}, \ldots, v_{\sigma_{|V|}}$ be the order of the vertices of V as they appear on $P_{|V|-3}$. The closable Hamilton path P in an edge-extension of G is constructed by connecting the vertices v_{σ_i} and $v_{\sigma_{i+1}}$ $(1 \le i < |V|)$. If $\{v_{\sigma_i}, v_{\sigma_{i+1}}\} \in E$, add $\{v_{\sigma_i}, v_{\sigma_{i+1}}\}$ to P. Otherwise draw an edge p from v_{σ_i} to $v_{\sigma_{i+1}}$ visiting only the faces that are also visited by $P_{|V|-3}$. Observe that each edge in E crossed by p is also crossed by $P_{|V|-3}$. Each time p crosses an edge $e \in E$, break e into two split edges and add a new vertex between these two edges. Also replace p by a path that traverses all these new vertices. Call the newly inserted edges of the path p *auxiliary edges* and add them to P.

Since $P_{|V|-3}$ is a simple path and each edge in E is crossed by only one edge in F, the construction of P can break each edge $\{u, v\}$ in E only into two split edges $\{u, v_{\text{new}}\}$ and $\{v_{\text{new}}, v\}$. Additionally, because of Invariant 2 the new vertex v_{new} is between u and v on P. Therefore P has the between property.

Corollary 8 *An edge-extension G^+ of a planar graph G and a closable Hamilton path P in G^+ can be found in linear time such that*

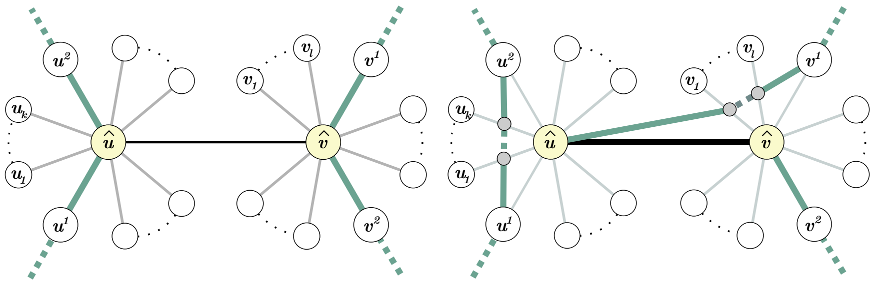
1. *each edge in G corresponds to a path of length two in G^+,*
2. *P has the between property,*
3. *each new vertex is incident to exactly two auxiliary and two split edges,*
4. *the auxiliary edges and split edges of each new vertex v_{new} alternate in the planar embedding of G^+ while turning around v_{new} and*
5. *the two auxiliary edges of each new vertex are part of P.*

4 Simultaneous embedding with fixed edges

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs and let $F \subset E_1 \cup E_2$. The goal is to find a simultaneous embedding of G_1 and G_2 such that the edges in F can be drawn in both embeddings as straight lines, in particular edges in $E_1 \cap E_2$ are drawn identically in the two embeddings. A first considered algorithm can handle only a very special set of fixed edges, more precisely, no vertex may be adjacent to more than one fixed edge. Later, this restriction is eased by a second algorithm.

Do for both graphs independently: Start searching for a Hamilton cycle C in an edge-extension of the considered graph as described in Section 3 and let φ be the combinatorial embedding used. Then, add the edges of F successively to the Hamilton cycle C.

Consider the situation shown in Figure 4. Let $\{\hat{u}, \hat{v}\} \in F$ be an edge that is not part of the Hamilton cycle. Since a Hamilton cycle contains all vertices, two other edges incident to $\hat{u}$ and $\hat{v}$, respectively, are part of the Hamilton cycle. Let E_v^e denote the sequence of edges incident to v in clockwise order around v in φ starting with the edge e. We add the edge $\{\hat{u}, \hat{v}\}$ to the Hamilton cycle in two steps.



(a) f not part of H (b) f part of H

Figure 4: A fixed edge f (black) and a ham. cycle H (bold).

Let $\{u^1, \hat{u}\}$ and $\{u^2, \hat{u}\}$ be the first and second edge in $E_{\hat{u}}^{(\hat{u},\hat{v})}$, respectively, that is part of the Hamilton cycle. Replace successively the edges $\{u_i, \hat{u}\}$ in the list $E_{\hat{u}}^{\{u^1,\hat{u}\}}$ between $\{u^1, \hat{u}\}$ and $\{u^2, \hat{u}\}$—but not equal to one of these—by a new vertex u_i^{new} and the edges $\{u_i, u_i^{\text{new}}\}$ and $\{u_i^{\text{new}}, \hat{u}\}$. Replace the part $u^1, \hat{u}, u^2$ of the Hamilton cycle by $u^1, \ldots, u_i^{\text{new}}, \ldots, u^2$. Vertex $\hat{u}$ is thus removed from the Hamilton cycle.

Let $\{v^1, \hat{v}\}$ and $\{v^2, \hat{v}\}$ be the first and second edge in $E_{\hat{v}}^{(\hat{u},\hat{v})}$, respectively, that is part of the Hamilton cycle. Replace successively the edges $\{v_i, \hat{v}\}$ in the list $E_{\hat{v}}^{(\hat{u},\hat{v})}$ between $\{\hat{u}, \hat{v}\}$ and $\{v^1, \hat{v}\}$—but not equal to one of these—by a new vertex v_i^{new} and the edges $\{v_i, v_i^{\text{new}}\}$ and $\{v_i^{\text{new}}, \hat{v}\}$. Replace the part $v^1, \hat{v}, v^2$ of the Hamilton cycle by $v^1, \ldots, v_i^{\text{new}}, \ldots, \hat{u}, \hat{v}, v^2$. Thus, the edge $\{\hat{u}, \hat{v}\}$ is part of the Hamilton cycle. Note that no edge in the "non-adjacent" set of edges F is ever split or removed from the Hamilton cycle and each new vertex still satisfies property 3-5 of Corollary 8.

Despite all these modifications, observe that the constructed graph is still an edge-extension of G and we can use the ideas of Section 2 to obtain a simultaneous embedding and to draw all edges in F as straight lines. However, the between property is lost by this modifications.

Using the ideas of Section 2 we get a simultaneous embedding, where all edges in F are embedded as straight lines. However, we do not know, how many bends are necessary for an edge outside the Hamilton cycle because an edge in the graph $G = (V, E)$ under consideration can correspond to a path of arbitrary length in the edge-extension G^+ of G.

The following lemma helps us to limit the number of bends per edge. Let $V_1 = V$ and V_2 be the set of new vertices of the edge-extension of G. Because of property 3-5 of Corollary 8 of the constructed Hamilton cycle, each vertex in V_2 is incident only to edges of the Hamilton cycle C and the considered path P. Thus, no further edge is split. However, for later purposes a more general statement is shown.

Lemma 9 *Let* $H = (V_1 \cup V_2, E)$ *be a planar graph,* C *a cycle in* H *that visits all vertices of* V_1*. Additionally, let* $\mathcal{P} = (v_1, v_2, \ldots, v_k)$ *be a path in* H *whose endpoints belong to* V_1 *and whose inner vertices all belong to* V_2*.* H *can be modified by adding edges and breaking some edges* $e \notin C \cup \mathcal{P}$ *incident to an inner vertex in* ≤ 3 *parts such that a cycle* $\hat{C}$ *can be found that visits all vertices of* V_1 *and such that* $\hat{C}$ *crosses* $\mathcal{P}$ *at most two times.*

A proof of Lemma 9 is given in the Appendix. Iterate Lemma 9 for each edge e.

Corollary 10 *Let* G *be a planar graph,* F *a set of edges and* G^+ *an edge-extension of* G *with Hamilton cycle* C *containing all edges in* F*. Another edge-extension* G^+_{new} *of* G *with Hamilton cycle* C_{new} *can be constructed such that* C_{new} *also contains all edges in* F *and each edge in* G *corresponds to a path of length* ≤ 3 *in* G^+_{new}*.*

Corollary 11 *A* 3*-bend or* 5*-bend simultaneous embedding of two planar graphs with a set of non-adjacent fixed edges can be found in time* $O(n)$ *depending on whether polynomial space is required or not.*

In the following the second algorithm is considered, which can handle a more general set of fixed edges.

Definition 12 (star-free) *For a given graph* $G = (V, E)$*, a set of edges* $F \subseteq E$ *is called star-free if* F *does not contain three or more edges that are incident to the same vertex.*

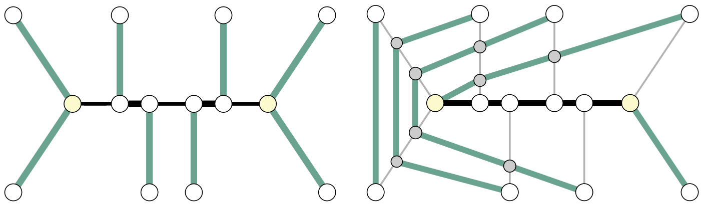
Definition 13 (cycle-free) *For a given graph* $G = (V, E)$*, a set of edges* $F \subseteq E$ *is called cycle-free if each cycle in* F *is a Hamilton cycle of* G*.*

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs and F a set of edges that is star- and cycle-free with respect to G_1 and G_2. The set F can now contain several paths of fixed edges. Let $P_1, \ldots, P_r$ denote all such paths with maximal length. Both graphs G_1 and G_2 are handled one after another. Again, using the ideas of Section 2, we need a Hamilton cycle C in an edge-extended graph G^+ that contains all the fixed edges. However, we have to add complete paths P_i to the Hamilton cycle.

This can be done iteratively for $i = 1, \ldots, r$. Take an arbitrary Hamilton cycle C_0 using the algorithm of Section 3 and let C_i be the Hamilton cycle after step i that contains all $P_1, \ldots, P_i$. It remains to show, how to add one path P_i to C_{i-1}. Use Lemma 9 to reduce the crossings of P_i and C_{i-1}. Edges incident to inner vertices of P_i are split ≤ 2 times. Handle the complete path P_i of fixed edges similarly to one fixed edge. Additionally, reroute the up to two crossings of C_{i-1} around one of the endpoints of P_i as shown in Figure 5. All edges incident to a vertex part of P_i are additionally split ≤ 2 times. Altogether, such an edge is split ≤ 4 times. In other words an edge is split in ≤ 5 parts.

Since an edge in G can be only incident to two inner vertices of paths $P_1, \ldots, P_r$, an edge can be split in $\leq 2 \cdot 4 + 1 = 9$ parts after iterating over all $P_1, \ldots, P_r$. Finally, we can again use Lemma 9 to reduce the $O(1)$ crossings to two of each edge and C_r without splitting further edges.

Corollary 14 *A 3-bend or 5-bend simultaneous embedding of two planar graphs with a star- and cycle-free set of fixed edges can be found in linear time depending on whether polynomial space is required or not.*



(a) P not part of H (b) P part of H

Figure 5: A path of fixed edges P (black) and some edges of H

5 A lower bound and other restrictions

Let us consider Figure 6 in order to confirm that there are triangulated planar graphs without a Hamilton cycle. Assume that the shown graph contains a Hamilton cycle. Since there are more white than black vertices, each Hamilton cycle must contain two consecutive white vertices. But this is not possible because none of the white vertices are adjacent.

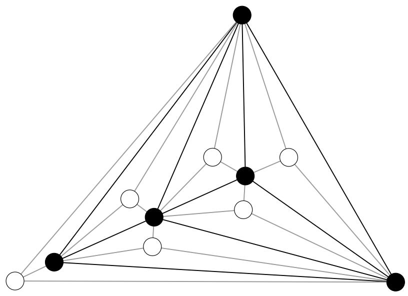
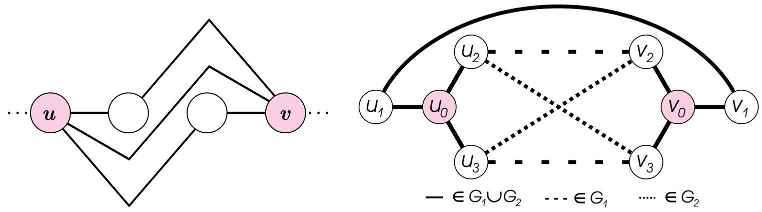


Figure 6: A triangulated graph without a Hamilton cycle.

Lemma 15 *No An embedding of a planar graph on a given set of points requires at least two bends at some edges.*

Proof: Let G be a planar, triangulated graph that has no Hamilton cycle. Assume that there is an embedding, where all points are on one line. Since only one bend is allowed, no edge $\{u, v\}$ as shown in Figure 7(a) can be used. Therefore and since G is triangulated, the adjacent points on the line must be connected by an edge. Again, since G is triangulated, an edge from the first to the last point on the edge must exist. Thus, this drawing contains a Hamilton cycle and is no embedding of G. □

The algorithm in the last section can only handle a star- and cycle-free set of fixed edges. Now the question arises whether this restriction is necessary or not. Let us consider first the case, where two triangulated planar graphs and a not cycle-free set of fixed edges are given. Let us denote this cycle of fixed edges by $C \subseteq F$. If there are two vertices



(a) All vertices on a line. (b) No simultaneous emb.

Figure 7: Two counterexamples.

not part of C that are on the same side of the cycle in one of the two graphs and that are on different sides in the other graph, no simultaneous embedding is possible. Second, consider two triangulated planar graphs and two vertices u_0 and v_0 that are incident to at least three fixed edges $\{u_0, u_1\}, \{u_0, u_2\}, \{u_0, u_3\}$ and $\{v_0, v_1\}, \{v_0, v_2\}, \{u_0, v_3\}$, respectively. See Figure 7(b). If in one graph the pairs of vertices $\{u_1, v_1\}$, $\{u_2, v_2\}$ and $\{u_3, v_3\}$, in the other graph pairs of vertices $\{u_1, v_1\}$, $\{u_2, v_3\}$ and $\{u_3, v_2\}$ are connected by vertex-disjoint paths, respectively, again no simultaneous embedding is possible.

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Appendix

Proof of Lemma 9: Consider a fixed combinatorial embedding of H. Consider the situation as shown in Figure 8 where $U = u_1, u_2, \dots$ and $W = w_1, w_2, \dots$ are two subsets of vertices of C. Let $l > 2$ be the number of crossings of C and $\mathcal{P}$. Thus, C consists of sub-paths $u_i, v_{\tau_{i,1}}, \dots, v_{\tau_{i,2}}, w_i$ $(1 \leq i \leq l)$. For an easier understanding, denote the sub-path $v_{\tau_{i,1}}, \dots, v_{\tau_{i,2}}$ by P_i. This gives us the imagination as if P_i is only one vertex. This is the right intuition because there are only two interesting edges incident to one endpoint of P_i.

Reduce the number of crossings of the path stepwise by replacing the first three crossings at P_1, P_2 and P_3 of C and $\mathcal{P}$ by only one crossing. The result is a slightly modified cycle $\overline{C}$ of C. Which vertex of u_2, u_3, w_1, w_2 and w_3 do we reach if we traverse C from u_1 not crossing P_1? The question can be answered only with u_2 or w_3. In the first case, C visits the vertices and sub-paths in the following order: $u_1, \dots, u_2, P_2, w_2, \dots, w_3, P_3, u_3, \dots, w_1, P_1$.

If required, break some edges incident to a internal vertex of $\mathcal{P}$ and add some new vertices between the split edge such that we can create a path $\overline{u_2, u_3}$ from u_2 to u_3 by adding further edges and vertices. Note that $\overline{u_2, u_3}$ is allowed to visit only the new vertices. Create a path $\overline{w_1, w_2}$ from w_1 to w_2 in the same way. In the modified graph we obtain $\overline{C}$ as $u_1, \dots, u_2, \overline{u_2, u_3}, u_3, \dots, w_1, \overline{w_1, w_2}, w_2, \dots, w_3 v_{\tau_{3,2}}, \dots, v_{\tau_{1,1}}$.

The second case is symmetric to the first case with a path from u_3 to w_1 part of the Hamilton cycle as there is a path from u_1 to w_3 part of the Hamilton cycle. Iterate this procedure always with the three vertices in U and W, respectively, whose indices are the smallest and that are still part of a crossing of $\overline{C}$ and $\mathcal{P}$. Finally, a cycle $\hat{C}$ in an edge-extension of H is found such that $\hat{C}$ crosses $\mathcal{P}$ only two times.

It remains to show that no edge is split more than twice while we connect two vertices in U or W. Since an edge is either split by connecting two vertices in U or by connecting two vertices in W, let us consider w.l.o.g. only what happens if vertices in U are connected with each other.

An edge adjacent to P_i is split once only iff a path from a vertex u_j to a vertex u_k with $j < i < k$ is created. Call a vertex (or a sub-path) of P excited if it is incident to an edge split once and call it finished if it is split twice.

During the iteration of the procedure above we can observe:

1. If P_i is excited, there is at most one P_j with $j < i$ part of the Hamilton cycle and
2. if P_i is finished, there is no P_j with $j < i$ part of the Hamilton cycle.

Since rerouting of the Hamilton cycle only breaks an edge adjacent to P_i once only an edge adjacent to P_i is split in ≤ 3 parts.

□

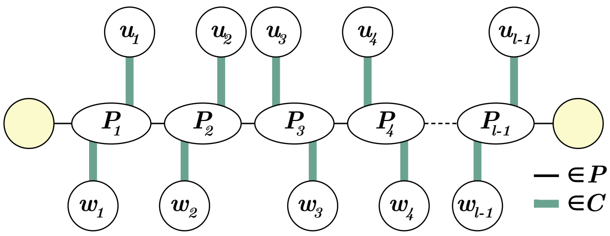

Figure 8: Crossing of a path and a Hamilton cycle