Discrete Geometry on Colored Point Sets in the Plane

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Let R, B and G denote disjoint three sets of red points, blue points and green points in the plane (on a line), respectively. The following theorem can be proved by using moment curve in the space and ham-sandwich theorem.

Theorem 1 (Kaneko, Kano and Watanabe (2015))**.** *Assume that R, B and G are on a line, $|R| = 2a$, $|B| = 2b$ and $|C| = 2c$, where a, b, c are positive integers. Let $I \cup J$ be a bipartition of $R \cup B \cup G$ by a point such that $|I| = |J| = a + b + c$. Then there exists a subset $X \subset I$ and $Y \subset J$ such that X and Y consist of consecutive points and both $I - X + Y$ and $J - Y + X$ are balanced, namely, they contain exactly a red points, b blue points and c green points, respectively (see Fig. 1).*

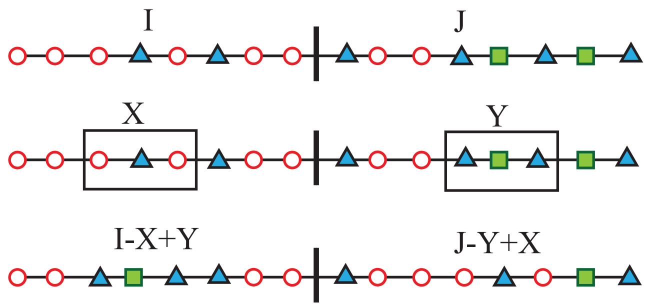

Figure 1: Bipartition $I \cup J$ and their subset X and Y given in Theorem 1

It is well-known that if n red points and n blue points are given in the plane in general position, then there exists a non-crossing geometric alternating perfect matching on these red and blue points, where each edge is a straight line segment and connects a red point and a blue point. Recently, this result is extended as follows.

Theorem 2 (Kano, Suzuki and Uno [6]). *Let $n \geq 1$ be an integer. If $|R \cup B \cup G| = 2n$ and $1 \leq |R|, |B|, |G| \leq n$, then there exists a non-crossing geometric properly colored perfect matching on $R \cup B \cup G$ (see Figure 2).*

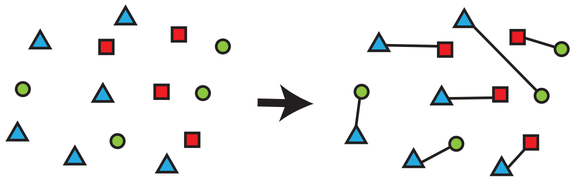

Figure 2: A non-crossing geometric properly colored perfect matching.

Theorems 2 is proved by using the following lemma.

Lemma 3. *Assume that R, B, G are on a line, $R \cup B \cup G$ contains even number of points, and the two end points have the same color. Then there exists a bipartition $I \cup J$ of $R \cup B \cup G$ by a point such that $|I|$ and $|J|$ are even and I and J satisfy $|I \cap X| \leq |I|/2$ and $|J \cap X| \leq |J|/2$ for every $X \in \{R, B, G\}$*

This lemma is recently extended to the plane as follows.

Theorem 4 (The hamburger theorem, Kano and Kynčl [5]). *Let $n \geq 2$ be an integer. If $|R| + |B| + |G| = 2n$ and $1 \leq |R|, |B|, |G| \leq n$, then there exists a line l such that each open half plane H defined by l satisfies that (i) $|(R \cup B \cup G) \cap H| \geq \min\{|R|, |B|, |G|\}$, (ii) $|(R \cup B \cup G) \cap H|$ is even, (iii) $0 \leq |R \cap H|, \ |B \cap H|, \ |G \cap H| \leq |(R \cup B \cup G) \cap H|/2$, and (iv) l passes through no point of $R \cup B \cup G$.*

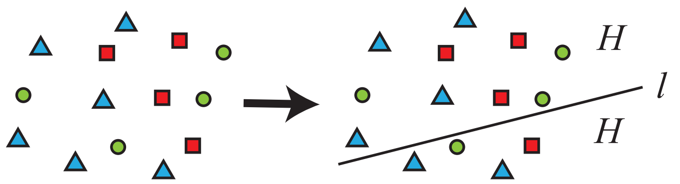

Figure 3: A balanced partition given in Theorem 4.

In the following theorem, a *2-fan* is defined to be a point x in the plane and two rays emanating from x.

Theorem 5 (Bárány and Matoušek [1]). *Any 3 measures can be simultaneously α-partition by a 2-fan for $\alpha = (1/2, 1/2)$ and for $\alpha = (2/3, 1/3)$.*

Theorem 6 (Bereg and Kano [3]). *Assume that $|R| = |B| = |G| = n \geq 2$ and all the vertices of the convex hull of $R \cup B \cup G$ have the same color. Then there exists a line that determines a half-plane containing exactly k red points, k blue points and k green pints for some integer $1 \leq k \leq n - 1$.*

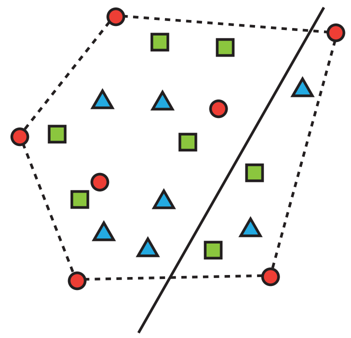

Figure 4: A balanced partition given in Theorem 6 with $n = 7$.

References

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